

Hermitian Analyticity, IR/UV Mixing and Unitarity of Noncommutative Field Theories

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ABSTRACT: The IR/UV mixing and the violation of unitarity are two of the most intriguing aspects of noncommutative quantum field theories. In this paper the relation between these two phenomena is explained and established. We start out by showing that the S -matrix of noncommutative field theories is hermitian analytic. As a consequence, a noncommutative field theory is unitary if the discontinuities of its Feynman diagram amplitudes agree with the expressions calculated using the Cutkosky formulae. These unitarity constraints relate the discontinuities of amplitudes with physical intermediate states; and allow us to see how the IR/UV mixing may lead to a breakdown of unitarity. Specifically, we show that the IR/UV singularity does not lead to the violation of unitarity in the space-space noncommutative case, but it does lead to its violation in a space-time noncommutative field theory. As a corollary, noncommutative field theory without IR/UV mixing will be unitary in both the space-space and space-time noncommutative case. To illustrate this, we introduce and analyse the noncommutative Lee model—an exactly solvable quantum field theory. We show that the model is free from the IR/UV mixing in both the space-space and space-time noncommutative cases. Our analysis is exact. Due to absence of the IR/UV mixing one can expect that the theory is unitary. We present some checks supporting this claim. Our analysis provides a counter example to the generally held beliefs that field theories with space-time noncommutativity are non-unitary.

KEYWORDS: Non-Commutative Geometry, Unitarity, Analyticity, S-Matrix .

*Supported by KBN grant 5P03B05620

Contents

1. Introduction

Recently there has been a lot of activities in constructing and understanding field theories on noncommutative spacetime (see e.g. [1, 2]). There are many reasons why such approaches are of interest, most of them related to the desire to take into consideration the quantum gravity effects and to understand the nature of spacetime at very short distances (see e.g. [3, 4]). Some of the most recently considered noncommutative geometries are the noncommutative Minkowski space $\mathbf{R}^{D-1,1}$ [5, 6, 7, 8, 9], the fuzzy sphere S_N^2 [10, 11, 12], and the κ -Minkowski spacetime [13, 14, 15]. The algebra of functions on noncommutative $\mathbf{R}^{D-1,1}$ is generated by noncommutative space-time coordinates \hat{x}^μ obeying the commutation relations ($\mu, \nu = 0, 1, \dots, D-1$).

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1.1)$$

where $\theta^{\mu\nu}$ is an anti-symmetric constant matrix. The fuzzy sphere S_N^2 is generated by Hermitian operators $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ satisfying the defining relations ($i, j, k = 1, 2, 3$).

$$[\hat{x}_i, \hat{x}_j] = i\lambda_N \epsilon_{ijk} \hat{x}_k, \quad \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = R^2. \quad (1.2)$$

Here the noncommutativity parameter λ_N has the dimension of length and should be taken positive. The radius R of the fuzzy sphere is quantized, in units of λ_N , by

$$\frac{R}{\lambda_N} = \sqrt{\frac{N}{2} \left(\frac{N}{2} + 1 \right)}, \quad N = 1, 2, \dots \quad (1.3)$$

The κ -Minkowski spacetime is defined by the basic relations between the three commuting space coordinates ($[\hat{x}_i, \hat{x}_j] = 0$) and a noncommutative quantum time variable \hat{t} ($\hat{x}_0 = c\hat{t}$):

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i. \quad (1.4)$$

In this paper we consider the case of noncommutative $\mathbf{R}^{D-1,1}$. This topic has been studied extensively (for a recent review, see e.g. [1, 2] and references therein). Field theory on this noncommutative space can be obtained by the replacement of standard products of fields by the Moyal $*$ -product induced by the relation (1.1) ¹,

$$A \cdot B(x) \longrightarrow A * B(x) = e^{-i\frac{\theta}{2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z''}} A(x + z') B(x + z'')|_{z'=z''=0}. \quad (1.5)$$

¹We denote $x = (\mathbf{x}, t)$, $z' = (\mathbf{z}', \tau')$, $z'' = (\mathbf{z}'', \tau'')$ and use the notation $a_\mu \theta^{\mu\nu} b_\nu \equiv a\theta b$.

In the momentum basis, the result of such an operation is the appearance of an additional Moyal phase factor $V(k^1, \dots, k^N)$

$$e^{ik^1x} * e^{ik^2x} * \dots * e^{ik^Nx} = V(k^1, \dots, k^N) e^{i \sum_i k^i x}, \quad V(k^1, \dots, k^N) := e^{\sum_{i \leq j} \frac{i}{2} k^i \theta k^j}. \quad (1.6)$$

Due to this phase factor one has to fix a definite cyclic ordering (say, anti-clockwise) of the momenta that enter any vertex of a given Feynman diagram.

An intriguing phenomenon for the quantum field theory on noncommutative $\mathbf{R}^{D-1,1}$ is the existence of an infrared/ultraviolet (IR/UV) mixing [16] in the quantum effective action. Due to this mixing, IR singularities arise from integrating out the UV degrees of freedom. This threatens the renormalizability and even the consistency of a QFT on noncommutative $\mathbf{R}^{D-1,1}$. Hence a better understanding (beyond the technical level) of the mechanism of IR/UV mixing and possible ways to resolve it are certainly highly desirable. We recall that so far in the literature, field theory on noncommutative $\mathbf{R}^{D-1,1}$ has been quantized by following the standard perturbative procedures: namely, the action is expanded around the free action and the corresponding Feynman rules are then written down. This is justified in the commutative case; however, since the introduction of $\theta^{\mu\nu}$ necessarily breaks the Lorentz symmetry from $SO(D-1, 1)$ to a smaller group that is left unbroken by the commutation relations (1.1), it is actually quite unnatural to employ the standard perturbative vacuum, i.e. the one defined by the free action and so respecting the *full* Lorentz symmetry. This leads one to suspect that the IR/UV mixing may be reflecting only the properties of the perturbation theory, and may be altered or disappear completely in the full nonperturbative regime (see for example, [17]). An exactly solvable field theory would be a good ground for testing this idea [18]. This leads us to introduce and study the noncommutative Lee model.

Another intriguing phenomenon for any quantum field theory on noncommutative spacetime is that unitarity could be violated. It is commonly believed that noncommutative field theory with space-space noncommutativity is unitary, while theory with space-time noncommutativity is not. This is consistent with the fact that space-space noncommutative field theory can be embedded in string theory [9, 19, 20, 21, 22, 23], while field theory with space-time noncommutativity cannot [24, 25, 26]. In [27] it was found that the unitarity constraints (see (2.11)) are satisfied for noncommutative theories with space noncommutativity but are violated for theories with a noncommuting time (see also [28, 29, 30, 31] for recent discussions). However these constraints are, in general, actually a stronger statement than the unitarity itself. The constraints presume a symmetric condition (see (2.12)) which is not generally valid. Without making any additional assumptions, in this paper, we examine directly the analyticity and unitarity of the S -matrix of a general noncommutative field theory. We show that Feynman amplitudes of a noncommutative theory are hermitian analytic (see (2.6)), a useful characterization of the S -matrix as introduced and proven by Olive [32]. As a result, the statement that the S -matrix is unitary takes the boundary-analytic form (2.7); and that the discontinuity of a Feynman diagram amplitude can be computed according to the Cutkosky formulae [33].

Although these two phenomena have received a lot of attention and have been thoroughly discussed in the literature, as far as we know, the relation between them has not been identified explicitly and explained before. One of the main aims of this paper is to identify and explain such a relation between IR/UV singularity and the possible violation of unitarity in a noncommutative field theory. This relation will be established through the boundary-analytic unitarity constraints (2.7). The basic idea is that the unitarity constraints allow one to relate the discontinuity of a scattering amplitude in a physical region with the appearance of intermediate states that can be put on-shell in this region. However, in a noncommutative theory, IR singularities can also be generated due to the IR/UV mixing. These new singularities do not correspond to any physical intermediate degrees of freedom. So, generally, one can expect that the unitarity constraints could be violated. In this paper we show, that in the case of space-space noncommutativity, the new IR singularities are safe in the sense that they do not generate any discontinuities in the scattering amplitudes. However, the IR singularities do generate such discontinuities in the space-time noncommutative case. This is the basic field theoretic mechanism for the violation of unitarity in a noncommutative theory. We stress that this violation of unitarity occurs only if time is noncommuting *and* in the presence of singularities due to the IR/UV mixing.

To illustrate the above ideas, we introduce and analyse the noncommutative Lee model. Lee model [34] is an exactly solvable, nonrelativistic model. The noncommutative Lee model can be defined by using the deformed product of fields (1.5). The model remains exactly solvable. We show that the noncommutative Lee model is free from the IR/UV mixing both at the perturbative level, and in the full exact answer. Thus the noncommutative Lee model does not provide a resolution of the IR/UV mixing issue. This may appear to be disappointing from the point of view of looking for a nonperturbative resolution of the IR/UV mixing issue. Nevertheless, the absence of an IR/UV singularity in a noncommutative field theory is nontrivial. This is one of the main results of this paper. Moreover, due to the absence of the IR/UV mixing, one can expect, from the the above mentioned general arguments, that the Lee model with space-time noncommutativity is unitary. We provide some further arguments to support this claim.

The plan of our presentation is as follows: In section 2.1, we review some basic facts about the S -matrix of commutative field theory. In section 2.2, we prove that Feynman diagram amplitudes in a noncommutative field theory are hermitian analytic and we investigate the consequences of this statement on the unitarity of the theory. We show that the usual form of the unitarity constraints used by many people is not correct in general. We derive the correct form of the unitarity constraints and show how they can be used to check the unitarity of a given noncommutative theory. In section 2.3, we explain how a IR/UV singularity may lead to a breakdown of unitarity in space-time noncommutative field theory. In section 3, we study the issue of the IR/UV mixing and unitarity in the noncommutative Lee model. In section 3.1 we

describe the commutative Lee model. We show that this model is renormalizable with the renormalization constants easily computed in a closed form. It is well known that the original Lee model in 4-dimensional spacetime has a ghost state and is not unitary [34, 35, 36]. We discuss improved versions of the original Lee model that do not have these problems; and restrict ourselves to these models when we introduce noncommutativity and address the issue of the unitarity of the noncommutative model. This we do in section 3.2 where we introduce the space-space noncommutative and the space-time noncommutative Lee model via the substitutions (3.33) and (3.34). We show that there is no IR/UV mixing in either case and one can expect that the theory is unitary. We present some arguments supporting this claim.

2. Unitarity and Hermitian Analyticity

In this section, we discuss some useful properties of the S -matrix. We refer the reader to [37] and to the excellent monograph [38] for further details on this subject. We follow the notations and nomenclature of [38].

2.1. S -Matrix in the Commutative Case

First we consider the commutative case. Unitarity of a quantum field theory follows from the existence of a hermitian Hamiltonian. In terms of the onshell S -matrix, unitarity is the statement that

$$SS^\dagger = S^\dagger S = 1. \quad (2.1)$$

Due to the cluster decomposition property of the S -matrix, it is meaningful to decompose S into two parts

$$S = 1 + iT, \quad (2.2)$$

with T is the transition matrix. Written in terms of $T_{ab} := \langle a|T|b\rangle$, we have

$$T_{ab} - T_{ba}^* = i \sum_n T_{na}^* T_{nb} = i \sum_n T_{an} T_{bn}^*, \quad (2.3)$$

where the sum is over all intermediate states associated with putting particles onshell. The S -matrix and the transition matrix T are defined for external particles with real momenta. Since both are invariant under proper Lorentz transformations their matrix elements (transition amplitudes) must be functions of Lorentz scalars which can be formed out of the momenta. We call a combination of external lines of the amplitude for a given physical process a *channel*, and two channels whose lines are disjoint and exhaustive a *reaction*. For an amplitude with n external lines, there are $2^{n-1} - n - 1$ different reactions provided that we exclude reactions with single-particle channels and do not distinguish the direction of the reaction. The channel invariant variable is the square of the energy in the given channel C ,

$$s = s_C = -\left(\sum_{i \in C} \pm p_i\right)^2 \quad (2.4)$$

where $\pm p_i$ are the momenta of incoming and outgoing lines, respectively. s_C 's are generalizations of the Mandelstam s, t, u variables for $2 \rightarrow 2$ scattering. It is convenient to discuss the singularity structure of a scattering amplitude in terms of the space of these $2^{n-1} - n - 1$ different channel invariants. For more details see: [37].

The transition amplitudes typically have singularities. In perturbation theory, the transition amplitude T_{ab} is given by the sum of a number of Feynman diagrams M_{ab} , each corresponding to a different channel. The Feynman integral is typically of the form

$$I_G(p) = \int \prod_l^L d^D k_l \prod_i^I \frac{i}{q_i^2 - m_i^2} \cdot B, \quad (2.5)$$

where B is a real normalization factor that contains the couplings and factors of π, i etc and p 's are the external momentum. As we have said before, the integral can be written in terms of the s 's. If one extends s to the complex plane, then the singularities are typically branch points in the complex s -plane². Extending s to the complex domain, one can think of T_{ab} (or M_{ab}) as the boundary value of an analytic function defined on the complex s -plane. The resulting analytic function has singularities on the real s -axis that correspond to physically accessible momenta. These singularities are called the *physical region singularities*. In addition, this analytic function may have additional singularities that correspond to external momenta that are not physically accessible. The analysis of these additional singularities is more complicated and is not usually performed.

The existence of singularities in the amplitude is a consequence of unitarity [32]. The reasoning is that as the channel invariant increases past a certain threshold (in the physical region of the considered amplitude) that corresponds to a new possible intermediate state, a new term enters the unitarity equation and this gives rise to a singularity in that channel. Such singularities are called *normal thresholds*. The physical region is divided into segments by the normal thresholds singularities. It can be shown, within perturbation theory, that the amplitudes in these segments can be continued consistently into the complex plane and be related analytically if one adopts in the Feynman integrals the $+i\epsilon$ prescription by replacing $m^2 \rightarrow m^2 - i\epsilon$, $\epsilon > 0$. This corresponds to associating an $+i\epsilon$ with a channel invariant when it is close to a normal threshold. The $+i\epsilon$ prescription in the correct invariant is appropriate for all physical region normal thresholds in all amplitudes [38]. Furthermore it can be shown that the Feynman amplitudes (and hence also T) are *hermitian analytic* [32], i.e. they satisfy:

$$M_{ab}(s)^* = M_{ba}(s^*). \quad (2.6)$$

²The locations of the singularities are determined by the Landau equations, see for example [38]. We remark that the Landau equations are entirely fixed in terms of the singularity manifold \mathcal{T} of the *integrand* of the Feynman integral, and since noncommutativity modifies the integrand by a phase factor, the Landau equations are unmodified by noncommutativity.

As a consequence of the hermitian analyticity (2.6), the unitarity relation (2.3) can be put in a more elegant form

$$\text{Disc } T_{ab} = i \sum_n T_{an}^{(-)} T_{nb}^{(+)} = i \sum_n T_{an}^{(+)} T_{nb}^{(-)}. \quad (2.7)$$

Here $f^{(\pm)}$ denotes the boundary values, on the real axis, respectively from above and below the cut, of a complex function f ,

$$f^{(\pm)}(s) := \lim_{\epsilon \rightarrow 0^+} f(s \pm i\epsilon), \quad s \in \mathbf{R}, \quad (2.8)$$

and $\text{Disc } f$ is the discontinuity across this cut

$$\text{Disc } f := f^{(+)} - f^{(-)}. \quad (2.9)$$

The relation (2.7) is actually somewhat stronger. Indeed, as a result of unitarity and hermitian analyticity, it holds for each individual Feynman diagram [33]

$$\text{Disc } M_{ab} = i \sum_n M_{an}^{(-)} M_{nb}^{(+)} = i \sum_n M_{an}^{(+)} M_{nb}^{(-)}. \quad (2.10)$$

In (2.7) and (2.10) the discontinuities in a given channel of the amplitude are associated with normal thresholds.

In terms of Feynman diagrams, the matrix elements $M_{ab}^{(\pm)}$ are given, respectively, in terms of the $\pm i\epsilon$ prescription: $m^2 \rightarrow m^2 \mp i\epsilon$. The RHS of (2.10) can be computed using the “cutting rules” of Cutkosky [33]: first cut the diagram in all possible ways such that the cut propagators can go on shell simultaneously (for a given set of s ’s), then, for each cut, replace the propagators by $-2\pi i \delta(p^2 - m^2)$ in the relativistic case, and by $-2\pi i \delta(p_0 - E(\mathbf{p}, m))$ in the nonrelativistic case. Finally sum the contributions of all possible cuts.

Before we embark on the noncommutative case, let us remark that the equation (2.3) is sometimes written in the form [39] (or for M),

$$2 \text{Im} T_{ab} = \sum_n T_{na}^* T_{nb}. \quad (2.11)$$

To arrive at this form, the following symmetric relation

$$T_{ab} = T_{ba} \quad (2.12)$$

has been assumed. This relation holds, for example, when the theory is T -invariant and rotationally invariant, and the basis vectors $|a\rangle$ are chosen to be eigenstates of the total angular momentum [40]. However, we would like to stress that this relation is not true in general. Failure of (2.11) can be due to either the symmetry condition (2.12) or the unitarity of the theory (2.3) not being satisfied or if the amplitude possesses singularities which are not due to the possible intermediate states. Therefore, generically, (2.11) is not a conclusive check of whether a given theory is unitary or not. In the next subsection we show that the hermitian analyticity remains valid in the noncommutative case and, therefore, that (2.7) and (2.10) can be used to check unitarity of a noncommutative theory.

2.2. S -Matrix in the Noncommutative Case

In a noncommutative quantum field theory the propagators take the same form as in the commutative case while the vertices are modified by the Moyal phase factor (1.6) that arises from the noncommutative multiplication. For example, in the noncommutative ϕ^3 model, the modification of the (real) coupling is a multiplication by a real factor

$$g \rightarrow g \cos\left(\frac{1}{2}p\theta k\right), \quad (2.13)$$

where k and p are the momenta entering the vertex. However, it is easy to see that when the theory involves more fields, the modification of the vertex is, generally, a phase factor. For example, this is the case for the noncommutative Lee model to be introduced in the next section. The phase factor (1.6) is cyclically symmetric but not permutation symmetric. Therefore, the symmetric relation is, in general, not valid.

Since Lorentz invariance is broken, in addition to the channel invariants we have introduced above, the S -matrix of a noncommutative field theory generally depends also on the variables

$$\tilde{s}_C = -\left(\sum_{i \in C} \pm \tilde{p}_i\right)^2, \quad \tilde{p}_i := \theta p_i. \quad (2.14)$$

A novelty in noncommutative theory is the possible existence of the IR/UV mixing [16], which states that the amplitudes in a noncommutative theory become singular in the $\tilde{s} = 0$ limit as one removes the cutoff, i.e. $\Lambda \rightarrow \infty$. These singularities occur in the physical region of momenta but do not correspond to normal thresholds since the IR/UV singularities are not related to any new degrees of freedom. One may extend the amplitude analytically to above the cut associated with these singularities by adding $+i\epsilon$ to \tilde{s} . This corresponds to extending the $i\epsilon$ prescription for the Feynman diagram to the cutoff: $\Lambda^2 \rightarrow \Lambda^2 + i\epsilon$ since the combination $1/\Lambda^2 - \tilde{s}$ often appears together [16].

Hermitian analyticity

Next we examine the hermitian analyticity of a noncommutative Feynman diagram. We show that the Feynman amplitudes for noncommutative theories are hermitian analytic. To see this, we note that under the complex conjugation, the Moyal phase factor (1.6) becomes

$$V(k^1, k^2, \dots, k^N)^* = V(k^N, \dots, k^2, k^1), \quad (2.15)$$

i.e. it reverses the cyclic ordering of the momenta entering the vertex. We can interpret the RHS as the Moyal phase factor of a vertex which is the mirror image of the original one, see figure 1. In the operator language the RHS of (2.15) corresponds to a Wick contraction in the reverse order. For example,

$$\begin{aligned} M_{ab} &\sim \langle 0 | a_1 a_2 (\bar{\phi}_1 * \bar{\phi}_2 * \phi_3) a_3^\dagger | 0 \rangle \sim V(k_1, k_2, k_3) \\ M_{ba} &\sim \langle 0 | a_3 (\bar{\phi}_3 * \phi_2 * \phi_1) a_1^\dagger a_2^\dagger | 0 \rangle \sim V(k_3, k_2, k_1) = V(k_1, k_2, k_3)^*, \end{aligned} \quad (2.16)$$

where $|a\rangle = a_1^\dagger(k_1)a_2^\dagger(k_2)|0\rangle$, $|b\rangle = a_3^\dagger(k_3)|0\rangle$ in this example. In general, let

$$V^G := \prod_{v \in G} V_v \quad (2.17)$$

be the product of the Moyal phase factors associated with the vertices v of a Feynman diagram G . We have

$$(V^G)^* = V^{\bar{G}}, \quad (2.18)$$

where \bar{G} is the mirror diagram of G .

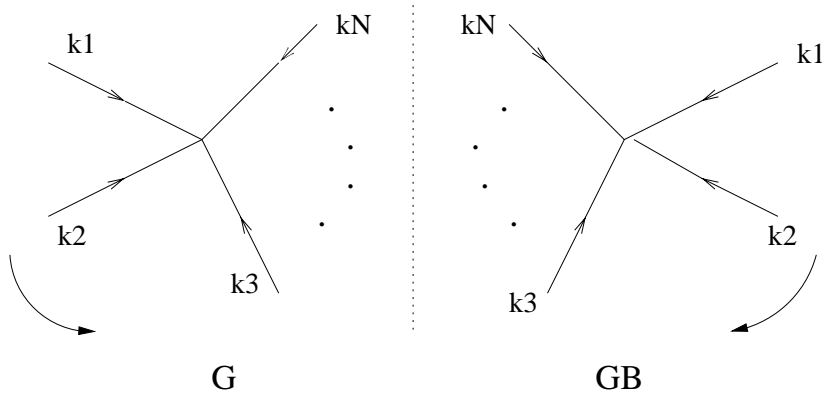


Figure 1: A Feynman diagram G and its mirror diagram \bar{G} . They have the opposite Moyal phase factors.

In a noncommutative theory, the Feynman amplitude for a diagram G takes the form

$$M_{ab}^G(s, \tilde{s}) = \prod_{l,i} \int \frac{d^D k_l B}{D_i^+} V^G. \quad (2.19)$$

Here $1/D_i^\pm$ is the propagator of the i -th internal line and the mass square has a small \mp imaginary part and B is a real normalization factor that contains the couplings³ and factors of π , i etc. Complex conjugating, one has

$$(M_{ab}^G(s, \tilde{s}))^* = \prod_{l,i} \int \frac{d^D k_l B}{D_i^-} (V^G)^* = M_{ba}^{\bar{G}}(s^*, \tilde{s}^*). \quad (2.20)$$

where we have used in the last step the observation that a change of sign in the imaginary part of the mass (or cutoff) corresponds to the change of sign in the imaginary part of s (or \tilde{s}). In the discussion given above, for the clarity of the argument, we have been careful to indicate which diagram (G or \bar{G}) is to be drawn for the Feynman amplitude to be computed. However this is not really necessary as which diagram has

³We emphasize that the couplings (bare as well as the renormalized one) have to be real. As we discuss at the end of section 3.1, the original Lee model (defined in 4-dimensional spacetime and with the dispersion relations (3.2)) has an imaginary bare coupling [34] and Hermitian analyticity does not hold, in both the commutative and noncommutative cases. However, the improved Lee models have real couplings and so have hermitian analytic S -matrix.

to be drawn is already clear once the process to be considered ($a \rightarrow b$ or $b \rightarrow a$) is specified. Therefore, can simply write

$$(M_{ab}(s, \tilde{s}))^* = M_{ba}(s^*, \tilde{s}^*). \quad (2.21)$$

Thus we have shown that the Feynman diagrams (and hence the S -matrix) of a non-commutative theory are hermitian analytic. We stress that our result is general and does not depend on the detailed form of the propagators or vertices. For example, it applies to the noncommutative Lee model to be introduced in section 3.

2.3. Unitarity Constraints and their Relation to the IR/UV Singularities

Note that the symmetric condition (2.12) is, in general, not valid and so the condition (2.11) may not hold even if a theory is unitary. However, since Feynman amplitudes satisfy hermitian analyticity, (2.7) and (2.10) hold if the S -matrix is unitary. Therefore we propose to use (2.7) or (2.10) instead of (2.11)⁴ as a check of unitarity.

Before we consider a specific model, let us discuss how the IR/UV singularities may lead to a breakdown of unitarity in general. Generally, a new IR/UV singularity in a scattering amplitude can be a pole or a branch point in $\tilde{s} = 0$, for some \tilde{s} . Note that

$$\tilde{s} = (\theta_E)^2(p_0^2 - p_1^2) + (\theta_B)^2(p_2^2 + p_3^2), \quad (2.22)$$

where we have chosen, for example, $\theta^{01} = \theta_E$, $\theta^{23} = \theta_B$ with all other components vanishing. Therefore for space noncommutativity, \tilde{s} is positive definite and so there is no new contribution to the discontinuity of the amplitude from this singularity. However, in the case of space-time noncommutativity, \tilde{s} is not of definite sign in the physical region [27]. Therefore if \tilde{s} is a branch point singularity, there will now be a new contribution to the LHS of (2.10). Since the IR/UV singularities do not correspond to any intermediate degrees of freedom that can go on shell, these new contributions will not be accounted for by the “onshell” sum and (2.10) will be violated. This is the basic mechanism how unitarity is violated by the IR/UV singularities when time is noncommuting.

3. An Application: The Noncommutative Lee Model

In this section, we consider the Lee model in D spacetime dimensions and its noncommutative generalization. In particular, we consider the issues of the IR/UV mixing and unitarity for the noncommutative Lee model. We will find that due to the presence of the Moyal phase factors the symmetric condition is not satisfied. Therefore

⁴In [27], the $1 \rightarrow 1$ propagator diagram in the noncommutative ϕ^3 and the $2 \rightarrow 2$ scattering diagram in the noncommutative ϕ^4 were considered. It is easy to see that the symmetric condition (2.12) is satisfied for these processes, and so checking of (2.11) constitutes a valid test of unitarity for the noncommutative theories considered there.

one should check unitarity using (2.10). We show that (and this result is exact) the noncommutative Lee model is free from any IR/UV singularity. As a result, one can expect that the noncommutative Lee model is unitary for both the space-space and space-time noncommutative case. We give further arguments supporting this claim.

3.1. Commutative Case

The Lee model was originally introduced by Lee in [34] where it was shown that the model is renormalizable with its mass, wavefunction and charge renormalizations easily performed in an exact manner. In the following, we follow the presentation of [40]. The model has two fermions V and N with masses $m_V^{(0)}, m_N^{(0)}$ respectively, and a real scalar φ with mass $m_\varphi^{(0)} := \mu_0$. The Hamiltonian for the free fields is:

$$H_0 = \int d^{D-1} \mathbf{p} \left[E_V(\mathbf{p}) V^\dagger(\mathbf{p}) V(\mathbf{p}) + E_N(\mathbf{p}) N^\dagger(\mathbf{p}) N(\mathbf{p}) + E_\varphi(\mathbf{p}) \varphi^\dagger(\mathbf{p}) \varphi(\mathbf{p}) \right], \quad (3.1)$$

where $E_V(\mathbf{p})$, $E_N(\mathbf{p})$, $E_\varphi(\mathbf{p})$ are the dispersion relations for the free V , N and φ particles, $N(\mathbf{p})$, $V(\mathbf{p})$ and $\varphi(\mathbf{p})$ are the annihilation operators of the N , V and φ particles, respectively. In the original Lee model [34], $D = 4$ and the fermions are taken to be very heavy while φ is assumed to be relativistic. In this case, the dispersion relations are given by

$$E_V = m_V^{(0)}, \quad E_N = m_N^{(0)}, \quad E_\varphi(\mathbf{k}) = (\mathbf{k}^2 + \mu_0^2)^{1/2} := \omega_{\mathbf{k}}. \quad (3.2)$$

The Galilei-invariant form [43]

$$E_A(\mathbf{p}) = \frac{\mathbf{p}^2}{2m_A^{(0)}}, \quad A = V, N, \varphi, \quad (3.3)$$

as well as the relativistic choice [44]

$$E_A(\mathbf{p}) = (\mathbf{p}^2 + m_A^{(0)2})^{1/2} \quad (3.4)$$

were also studied in the literature. The interacting Hamiltonian of the model is taken to be given by

$$H_{\text{int}} = g_0 \int \frac{d^{D-1} \mathbf{k}}{\sqrt{(2\pi)^{D-1}} 2\omega_{\mathbf{k}}} \int d^{D-1} \mathbf{p} \left(V^\dagger(\mathbf{p}) N(\mathbf{p} - \mathbf{k}) \varphi(\mathbf{k}) f(\mathbf{k}) + \varphi^\dagger(\mathbf{k}) N^\dagger(\mathbf{p} - \mathbf{k}) V(\mathbf{p}) f^*(\mathbf{k}) \right), \quad (3.5)$$

where $f(\mathbf{k})$ is a form factor⁵ introduced to smooth out the interaction to avoid the divergences connected with a point interaction. In fact f can be taken to be $f = 1$

⁵Note that, in principle, one can also use a more general form factor f that depends on the momentum of the $N\varphi$ pair. It is easy to see that this amounts to a simple replacement

$$f(\mathbf{k}) \longrightarrow f(k, p). \quad (3.6)$$

in the analysis below.

and the divergences can be absorbed by renormalization. This is the case of interest to us. However as we will see, the introduction of noncommutativity to the Lee model amounts to a modification of f by a phase factor. Therefore we will keep f explicitly in the presentation below, with the understanding that it will be set to 1 (or to the Moyal phase factor for the noncommutative case) in the final answer.

We note that the interaction H_{int} is nonlocal in space even in the limit $f = 1$. To see this, it is convenient to introduce the negative and positive frequency parts of φ :

$$\varphi(x) = a(x) + a^\dagger(x), \quad (3.7)$$

$$a(x) = \int \frac{d^{D-1}\mathbf{k}}{\sqrt{(2\pi)^{D-1} 2\omega_{\mathbf{k}}}} \varphi(\mathbf{k}) e^{ik \cdot x}, \quad a^\dagger(x) = \int \frac{d^{D-1}\mathbf{k}}{\sqrt{(2\pi)^{D-1} 2\omega_{\mathbf{k}}}} \varphi^\dagger(\mathbf{k}) e^{-ik \cdot x}. \quad (3.8)$$

In terms of a and a^\dagger , H_{int} can be written in the coordinate space as

$$H_{\text{int}} = g_0 \int d^{D-1}\mathbf{x} d^{D-1}\mathbf{y} \left(V^\dagger(\mathbf{x}, t) N(\mathbf{x}, t) \tilde{f}(\mathbf{x} - \mathbf{y}) a(\mathbf{y}, t) + N^\dagger(\mathbf{x}, t) V(\mathbf{x}, t) \tilde{f}^*(\mathbf{x} - \mathbf{y}) a^\dagger(\mathbf{y}, t) \right), \quad (3.9)$$

where \tilde{f} is the Fourier transform of the Lee model form factor and $\tilde{f} \rightarrow \delta(\mathbf{x})$ in the limit $f \rightarrow 1$. It is now clear that the coupling term is nonlocal in space since the operation of taking the positive frequency part involves the integration over all space. However the model is local in time.

Since the theory is local in time, it can be described equivalently in the Lagrangian formulation by performing the Legendre transformation. The Lagrangian density of the model is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad (3.10)$$

where \mathcal{L}_0 is the free part:

$$\mathcal{L}_0 = V^\dagger \left(i \frac{\partial}{\partial t} + E_V(-i\nabla) \right) V + N^\dagger \left(i \frac{\partial}{\partial t} + E_N(-i\nabla) \right) N + a^\dagger \left(i \frac{\partial}{\partial t} + E_\varphi(-i\nabla) \right) a, \quad (3.11)$$

and the interaction is described by

$$\mathcal{L}_{\text{int}} = g_0 \int d^{D-1}\mathbf{y} V^\dagger(\mathbf{x}, t) N(\mathbf{x}, t) \tilde{f}(\mathbf{x} - \mathbf{y}) a(\mathbf{y}, t) + H.C. \quad (3.12)$$

The Lagrangian formulation will be useful when we introduce an electric deformation of the model.

The Lee model can be solved by considering directly the Schrödinger equation with the Hamiltonian $H = H_0 + H_{\text{int}}$ where H_0 is given by (3.1) with the choice (3.2) and H_{int} is given by (3.5). Due to the structure of the interaction (3.12), the only elementary interaction of the theory involves the process

$$V \rightleftharpoons N + \varphi. \quad (3.13)$$

In a standard relativistic model, the antiparticle $\bar{\varphi}$ would appear and the crossed reaction

$$V + \bar{\varphi} \rightleftharpoons N \quad (3.14)$$

would be possible, but this is not allowed in the Lee model due to the particular form of the interaction Hamiltonian (3.5). The system possesses two simple conservation laws

$$n_V + n_N = \text{constant}, \quad n_V + n_\varphi = \text{constant}, \quad (3.15)$$

where n_V, n_N, n_φ are the total numbers of V, N, φ particles, respectively. Due to the conservation laws (3.15), the eigenfunctions of H contain only a finite number of particles and, consequently, the theory is exactly solvable [34].

Renormalization

The quantization of the theory is straightforward. Locality in time allows us to perform the standard canonical quantization of the theory. The nontrivial commutation relations of the field operators are

$$[\varphi(\mathbf{k}), \varphi^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'), \quad [N(\mathbf{p}), N^\dagger(\mathbf{p}')]_+ = \delta(\mathbf{p} - \mathbf{p}'), \quad [V(\mathbf{p}), V^\dagger(\mathbf{p}')]_+ = \delta(\mathbf{p} - \mathbf{p}'), \quad (3.16)$$

with the rest equal to zero. The vacuum of the theory $|0\rangle$ is defined by

$$N(\mathbf{p})|0\rangle = V(\mathbf{p})|0\rangle = \varphi(\mathbf{p})|0\rangle = 0. \quad (3.17)$$

It is easy to verify that

$$H_{\text{int}}\varphi^\dagger(\mathbf{k})|0\rangle = 0, \quad H_{\text{int}}N^\dagger(\mathbf{p})|0\rangle = 0; \quad (3.18)$$

thus we can take the φ and N -quanta as the physical particles (of masses μ and m_N , respectively) and identify $\mu = \mu_0$, $m_{N_0} = m_N$, and there is only the renormalization of the mass of V to be considered.

Without any loss of generality we consider the dispersion relation s (3.2) in order to study the renormalization of the theory. Consider the sector of the theory associated with one physical V -particle. Denote the physical V -particle as $|\hat{V}(\mathbf{p})\rangle$. Due to the conservation law (3.15), we have

$$|\hat{V}(\mathbf{p})\rangle = \sqrt{Z_V} \left(V(\mathbf{p})^\dagger |0\rangle + \int d^{D-1}\mathbf{k} \Phi(\mathbf{k}) N^\dagger(\mathbf{p} - \mathbf{k}) \varphi^\dagger(\mathbf{k}) |0\rangle \right) \quad (3.19)$$

with the wavefunction $\Phi(\mathbf{k})$ still to be determined. Here $|\hat{V}(\mathbf{p})\rangle$ is an eigenstate of H

$$H|\hat{V}(\mathbf{p})\rangle = m_V|\hat{V}(\mathbf{p})\rangle. \quad (3.20)$$

The normalization of $|\hat{V}(\mathbf{p})\rangle$ yields

$$1 = Z_V \left(1 + \int d^{D-1}\mathbf{k} |\Phi(\mathbf{k})|^2 \right). \quad (3.21)$$

Contracting (3.20) with $\langle 0|V(\mathbf{p}')$, one obtains

$$m_{V_0} + \frac{g_0}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} f(\mathbf{k}) \Phi(\mathbf{k}) = m_V. \quad (3.22)$$

On the other hand, contracting (3.20) with $\langle 0|N(\mathbf{q})\varphi(\mathbf{l})$, one obtains

$$(m_V - m_N - \omega_{\mathbf{k}})\Phi(\mathbf{k}) = \frac{g_0}{(2\pi)^{(D-1)/2}} \frac{f^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}, \quad (3.23)$$

which gives

$$\Phi(\mathbf{k}) = \frac{g_0}{(2\pi)^{(D-1)/2}} \frac{f^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}(m_V - m_N - \omega_{\mathbf{k}})}}, \quad \text{for } m_V < m_N + \mu, \quad (3.24)$$

$$\Phi(\mathbf{k}) = \frac{g_0}{(2\pi)^{(D-1)/2}} P \frac{f^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}(m_V - m_N - \omega_{\mathbf{k}})}}, \quad \text{for } m_V > m_N + \mu. \quad (3.25)$$

Note that eq. (3.24) corresponds to the case when the V particle is stable; i.e. it cannot spontaneously decay into an N and φ particle. The decay of the V particle is allowed in the case of eq. (3.25). The renormalized coupling can be obtained by requiring the scattering process

$$N + \varphi \rightarrow N + \varphi \quad (3.26)$$

to be nonzero in the limit $f \rightarrow 1$.

As a result, we obtain the following renormalization constants

$$Z_V^{-1} = 1 + \frac{g_0^2}{(2\pi)^{D-1}} \int \frac{d^{D-1}\mathbf{k}}{2\omega_{\mathbf{k}}} \frac{|f(\mathbf{k})|^2}{(m_V - m_N - \omega_{\mathbf{k}})^2}, \quad (3.27)$$

$$m_V = m_{V_0} + \frac{g_0^2}{(2\pi)^{D-1}} \int \frac{d^{D-1}\mathbf{k}}{2\omega_{\mathbf{k}}} \frac{|f(\mathbf{k})|^2}{(m_V - m_N - \omega_{\mathbf{k}})}, \quad (3.28)$$

$$g^2 = g_0^2 Z_V. \quad (3.29)$$

The integrals in (3.27) and (3.28) are generally divergent in the limit $f \rightarrow 1$. As usual, all the scattering amplitudes ($N\varphi - N\varphi$, $V\varphi - V\varphi$, $V\varphi - N\varphi\varphi$, $N\varphi\varphi - N\varphi\varphi$ etc.) become finite after we have performed the renormalization (3.27), (3.28) and (3.29)⁶.

We would like to add a couple of comments:

i) One can perform a path integral quantization of the theory and one obtains the Feynman rules given in figure 2. Using these Feynman rules, it is straightforward to show that the above results for the renormalization can also be obtained in the Lagrangian framework and are exact in perturbation theory. Later we will use these Feynman rules to study the noncommutative Lee model, particularly, in the time non-commuting case.

ii) In the original Lee model [34], $D = 4$ and the mass renormalization constant is linearly divergent while the wavefunction renormalization is logarithmically divergent. It has been shown that the different choices (3.3) (Galilean kinematics) and (3.4) (relativistic kinematics) of dispersion relations lead to finite renormalizations when $f \rightarrow 1$.

Unitarity and the ghost state

The relation (3.29) between the renormalized coupling g and the bare coupling g_0

⁶For example, in the sector $V\varphi - N\varphi\varphi$, the renormalized scattering amplitudes $V\varphi \rightarrow V\varphi$, $V\varphi \rightarrow N\varphi\varphi$ and $N\varphi\varphi \rightarrow N\varphi\varphi$ were studied in [41] and [42]

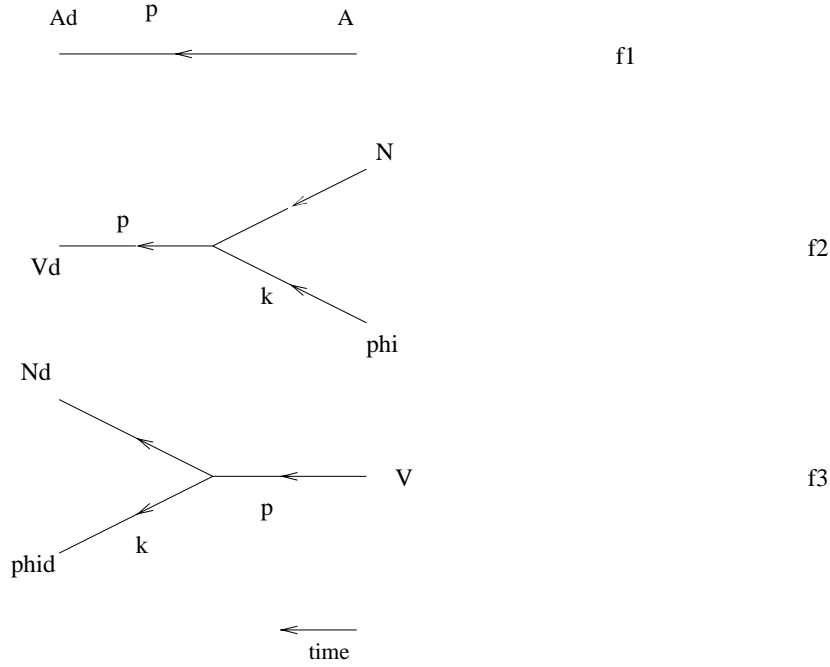


Figure 2: Feynman rules for the Lee model

can be rewritten as (with f set to 1)

$$g_0^2 = \frac{g^2}{1 - g^2 I}, \quad \text{where} \quad I \equiv \int \frac{d^{D-1} \mathbf{k}}{(2\pi)^{D-1} 2\omega_{\mathbf{k}}} \frac{1}{(m_V - m_N - \omega_{\mathbf{k}})^2} > 0. \quad (3.30)$$

For $D = 4$, I is logarithmically divergent. If g is to remain fixed and nonvanishing, the bare coupling has to be imaginary

$$g_0 = i\infty^{-1}. \quad (3.31)$$

and the wavefunction renormalization,

$$Z_V = 1 - g^2 I \rightarrow -\infty. \quad (3.32)$$

This contradicts the interpretation of Z_V as the probability of finding a bare V quantum in the physical V -particle state. Such negative probabilities imply that the S -matrix is not unitary. In fact one can show that [35] $Z_V < 0$ corresponds to a new state in the theory. This state $|G\rangle$ has a negative norm and is referred to as the “ghost state” by Kallen and Pauli. As a result, the S -matrix is explicitly non-unitary. In fact, the not unitarity of the theory is related to the original Hamiltonian being non-Hermitian due to the presence of an imaginary bare coupling.

Two improvements of the original Lee model are possible. One is to consider other dispersion relations e.g. (3.3) and (3.4). This leads to 4-dimensional theory with finite renormalizations and without a ghost [43, 44]. Another possibility is to consider the Lee model in lower dimensions [45]. In $D = 3$, the integral I in (3.30) is finite and so

the model is ghost free for physical coupling $0 < g < 1/\sqrt{I}$. The improved Lee model is still exactly solvable in both cases. To minimize the number of new formulae, we consider the second class of models when we generalize to the noncommutative case.

3.2. The Noncommutative Lee Model

The noncommutative framework is generated by using the \ast -product (1.5). As mentioned in the introduction, the noncommutative deformation can be introduced either in the Hamiltonian or the Lagrangian formulation in the magnetic case ($\theta^{i0} = 0$). The replacement (1.5) amounts to the following substitution in the formula (3.5):

$$f(\mathbf{k}) \longrightarrow f(k, p) := f(\mathbf{k}) e^{\frac{i}{2} p_i \theta^{ij} k_j}. \quad (3.33)$$

In the electric case with nonvanishing components $\theta^{i0} \neq 0$ ⁷, the substitution takes the form

$$f(\mathbf{k}) \longrightarrow f(k, p) := f(\mathbf{k}) e^{\frac{i}{2} \theta^{0i} (p_0 k_i - p_i k_0)}. \quad (3.34)$$

Obviously the \ast -product involves an infinite number of time derivatives. The nonlocalities in time destroy not just the usefulness of the Hamiltonian formulation, but also the standard way of relating the Lagrangian and the Hamiltonian description⁸. We are thus left only with the Lagrangian framework. For example, when there is only the nonvanishing component $\theta^{01} = \theta \neq 0$, one obtains the modification of the product of V and φ fields

$$V(\mathbf{x}, t) \ast a(\mathbf{y}, t) = V(\mathbf{x}, t) e^{-i\frac{\theta}{2} \left(\overleftarrow{\frac{\partial}{\partial t}} \frac{\partial}{\partial y_1} - \overleftarrow{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial t'} \right)} a(\mathbf{y}, t')|_{t=t'} = V(\mathbf{x}, t - \frac{i\theta}{2} \frac{\partial}{\partial y_1}) a(\mathbf{y}, t + \frac{i\theta}{2} \overleftarrow{\frac{\partial}{\partial x_1}}) \quad (3.35)$$

in the interaction Lagrangian (3.12). Note that due to the associativity of the Lagrangian and the integration over spacetime, the \ast -product of three fields in (3.12) can be represented by a modification of the product for any pair of fields ($V\varphi$ as in (3.35), VN or $N\varphi$).

Note also that the phase factor in (3.33) and (3.34) does not lead to a real factor as in the noncommutative scalar ϕ^3 case. Thus the noncommutative modification in the Lee model involves a complex factor. This, in particular, implies that the symmetric condition (2.12) is not satisfied.

Quantization of the magnetically deformed theory can be achieved by using either the canonical quantization, or equivalently a path integral quantization. In the electric case, canonical quantization fails due to the nonlocality in time. Nevertheless, formally, the theory can be quantized using the path integral method. In the following, we will

⁷Besides magnetic and electric cases one can also consider lightlike deformations [46], corresponding to the case $\theta_{\mu\nu} \theta^{\mu\nu} = 0$.

⁸For recent efforts at introducing a Hamiltonian framework for Lagrangian densities nonlocal in time see [47, 48, 49]. We have not been able to employ these results here in a constructive way.

use the path integral method to analyze both the magnetic and the electric Lee models. The Feynman rules are those of figure 1 with $f(k, p)$ given by (3.33) and (3.34) and work for general D . To be specific, below we consider the noncommutative Lee model in $D \leq 4$ dimensional spacetime and with the standard dispersion relations (3.2).

Renormalization and (no) IR/UV mixing

Since the effect of noncommutativity is a modification (3.33) or (3.34) of f by a phase factor, it is clear that the mass, wavefunction and coupling renormalization (depending on $|f|^2$) are not affected. Thus we conclude that the renormalization constants of the noncommutative Lee model are exactly computable and are independent of the noncommutativity parameter θ .

Moreover, one can easily convince oneself that the UV-divergences of the theory reside in planar diagrams that simply do not have nonplanar counterparts. Thus the UV-divergences of the noncommutative Lee model remain untouched in the limit when the cutoff is removed. This is quite different from the other noncommutative field theories which display an intriguing mixing of IR/UV [16]. In these models, the introduction of a nonzero noncommutativity improves the UV convergence of nonplanar diagrams but also leads to new IR singularities for these diagrams. In the present case of the noncommutative Lee model, there simply are no UV-divergences in the nonplanar diagrams, and hence there are also no new IR singularities that could be generated. We conclude that the noncommutative Lee model is *free from IR/UV mixing*. This result is exact.

Unitarity

First we consider the unitarity constraints at the one loop level. Due to the structure of the vertices (figure 2) in the theory, it is easy to convince oneself that only planar diagrams can be drawn at the one loop level. Therefore the one loop Feynman amplitudes take the form

$$M_{ab} = M_{ab}^{(0)} e^{i\phi_{ab}}, \quad (3.36)$$

where $M_{ab}^{(0)}$ are the corresponding amplitudes in the commutative case, and $e^{i\phi_{ab}}$ is the Moyal phase factor associated with the planar diagram. As a result, the equation (2.10) is satisfied since

$$\text{Disc} M_{ab} = e^{i\phi_{ab}} \text{Disc} M_{ab}^{(0)} = ie^{i\phi_{ab}} \sum_n M_{an}^{(0)} M_{nb}^{(0)} = i \sum_n M_{an} M_{nb}. \quad (3.37)$$

In the second step, we have used the fact that the constraint (2.10) is satisfied for the commutative Lee model since this model is unitary (or one can verify this in a straightforward manner since the M 's that appear in the sum are tree level ones). In the last step we have used the fact that the planar Moyal phase factor of the 1-loop diagram decomposes simply into the product of factors of the tree level ones:

$$e^{i\phi_{ab}} = e^{i\phi_{an}} e^{i\phi_{nb}}. \quad (3.38)$$

Note that due to the form of the modification for the one-loop amplitude (3.36), checking the imaginary part (2.11) would lead to the incorrect conclusion that the noncommutative Lee model is not unitary at a one loop level. Note also that the above argument is general and does not depend on whether θ is spacelike or timelike. Therefore, we conclude that the noncommutative Lee model is unitary at a one loop level for general $\theta^{\mu\nu}$. This result is valid to all orders in θ .

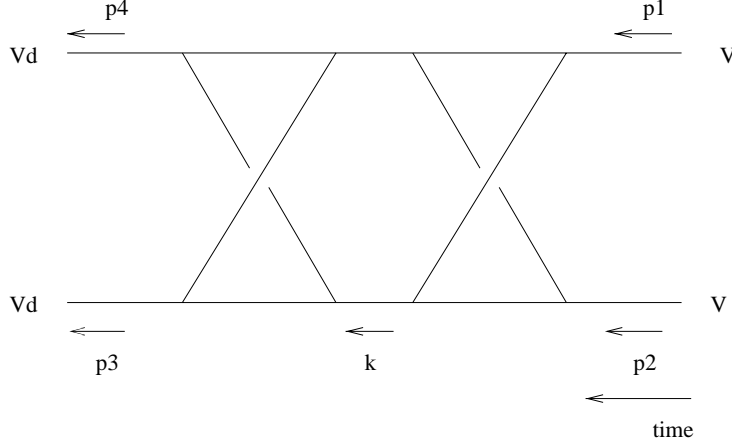


Figure 3: A nonplanar diagram

At a higher loop level, one can have nonplanar diagrams, for example, the one in figure 3. The phase factor associated with this diagram is

$$e^{\frac{i}{2}(p_1\theta p_2 - p_1\theta p_3 - p_2\theta p_3)} e^{-ik\theta(p_2 - p_3)}. \quad (3.39)$$

The second phase factor depends on the loop momentum and is a characterization of a nonplanar diagram. As one can check easily, this amplitude is regular in the variable \tilde{s} (and hence θ). Generally, due to the absence of the IR/UV singularity, a nonplanar amplitude will be regular in the variable \tilde{s} and so there is no new discontinuity in the LHS of the unitarity equation (2.10). Since both the LHS and RHS are regular in θ , the unitarity constraint will be satisfied at the zeroth order in θ . Although we believe this to be the case, it may not be easy to verify the unitarity relations to all orders in θ as one would have to exploit various nontrivial relations among special functions and integrals. The fact that unitarity constraints are satisfied at a one loop level; and also (at the zeroth order in θ) for any higher loop amplitude, is already a nontrivial property of the noncommutative Lee model. Without any other source of violation of unitarity in sight, we expect that the noncommutative Lee model is unitary for any $\theta^{\mu\nu}$.

4. Discussion

In this paper, we have discussed and examined two basic aspects of noncommutative field theories: the IR/UV mixing and unitarity. We have showed that the S -matrix

of a noncommutative field theory is hermitian analytic. This implies that unitarity provides a direct evaluation of the discontinuities associated with the cuts of normal thresholds. We have also explained how the IR/UV singularities can lead to a violation of unitarity for field theories with space-time noncommutativities. As a corollary, we have argued that a noncommutative field theory without any IR/UV mixing will be unitary in both the space-space and space-time noncommutative cases.

As an illustration of the general discussion, we have introduced and analysed the noncommutative Lee model. We have found that the model is entirely free from the IR/UV mixing. This result is exact. Our general arguments show that the noncommutative Lee model is unitary in both the space-space and space-time noncommutative cases. Simple explicit checks are consistent with this claim. Thus we provide a counter example to the general belief that field theories with space-time noncommutativity have to be non-unitary.

A consistent quantum field theory on a noncommutative spacetime should be unitary. It should also be free from the problems related to the IR/UV mixing. One can broadly divide the IR/UV mixing phenomena in noncommutative field theories into those that could be called good ones and bad ones. For example, the IR/UV singularities which appear in a purely bosonic noncommutative gauge theory or in a noncommutative QED are bad ones [50]. However, IR/UV singularities are milder and may be absent [51] in the presence of supersymmetry. The milder form of the IR/UV mixing in supersymmetric noncommutative gauge theories leads to a decoupling of the $U(1)$ degrees of freedom in the IR [52]. Not only the $U(1)$ degrees of freedom become free in the IR [52], they also trigger spontaneous supersymmetry breaking [53] in the presence of an appropriate Fayet-Iliopoulos D-term and play the rôle of the hidden sector. This we refer to as good IR/UV mixing effects. More details are provided in [54]. With unitarity better understood and (some) IR/UV mixing turned to be our advantage, it seems not unreasonable to contemplate that nature could indeed be noncommutative (at least at some level of explanation of its phenomena).

Acknowledgments

CSC and JL would like to thank Luis Alvarez-Gaume for useful discussions and comments and the theory group at CERN, where this work was started, for its hospitality. We would also like to thank David Fairlie, Pei-Ming Ho, Valya Khoze, Rodolfo Russo, Lenny Susskind, Richard Szabo and Gabriele Travaglini for helpful discussions.

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